Recent Advances on Robust Tensor Principal Component Analysis

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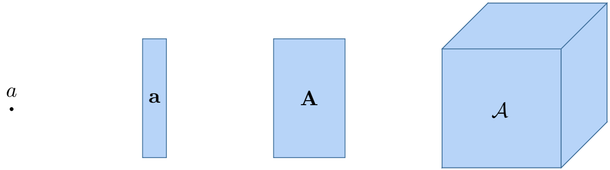
Outline

Definitions

- Tensors are multi-dimensional arrays, which are higher-order generalizations of matrices and vectors.



(a) scalar (b) vector (c) matrix (d) tensor

- Tube fibers and frontal slices of a third-order tensor

(a) Tube fibers $\mathcal{A}(i, j, :)$ (b) Frontal slices $\mathcal{A}(:, :, i)$ or $\mathcal{A}^{(i)}$

Tensor Multiplication

- For $\mathcal{A} \in \mathbb{R}^{I_1 \times P \times I_3}$ and $\mathcal{B} \in \mathbb{R}^{P \times I_2 \times I_3}$, define the **t-product**

$$\mathcal{C} = \mathcal{A} * \mathcal{B} \in \mathbb{R}^{I_1 \times I_2 \times I_3}$$

  which can be calculated by

$$\mathcal{C}(i_1, i_2, :) = \sum_{p=1}^{P} \mathcal{A}(i_1, p, :) \circledast \mathcal{B}(p, i_2, :),$$

  where $\circledast$ denotes circular convolution between two tube fibers.

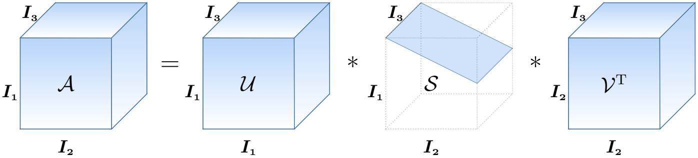
- Let $\hat{\mathcal{C}} = \mathrm{fft}[\mathcal{C}, [], 3]$ denote the result of fast Fourier transform (FFT) along the third mode of $\mathcal{C}$, the t-product can be calculated by matrix multiplication on each frontal slice separately:

$$\hat{\mathcal{C}}^{(i_3)} = \hat{\mathcal{A}}^{(i_3)} \times \hat{\mathcal{B}}^{(i_3)}, i_3 = 1, \cdots, I_3.$$

Tensor Singular Value Decomposition (T-SVD)

Let $\mathcal{A} \in \mathbb{R}^{I_1 \times I_2 \times I_3}$, $\mathcal{A}$ can be factored as

$$\mathcal{A} = \mathcal{U} * \mathcal{S} * \mathcal{V}^{\mathsf{T}}.$$



- $\mathcal{S}$ is the core singular value tensor and $*$ denotes t-product.
- $\mathcal{U}$ and $\mathcal{V}$ are orthogonal tensors, i.e. $\mathcal{U}^{\mathrm{T}} * \mathcal{U} = \mathcal{V}^{\mathrm{T}} * \mathcal{V} = \mathcal{I}$.
- In Fourier domain, $\hat{\mathcal{A}}^{(i)} = \hat{\mathcal{U}}^{(i)} \times \hat{\mathcal{S}}^{(i)} \times \hat{\mathcal{V}}^{(i)}$, $i = 1, \cdots, I_3$.
- The tensor nuclear norm (TNN) is defined as the average value of the matrix nuclear norm of all frontal slices in the Fourier domain.

Tensor Singular Value Decomposition: Algorithm

The following algorithm is for tensor singular value decomposition.

Algorithm 1: T-SVD for order-3 tensor

Input: $\mathcal{A} \in \mathbb{R}^{I_1 \times I_2 \times I_3}$.

$\hat{\mathcal{A}} \leftarrow$fft($\mathcal{A}$,[],3),

for $i_3 = 1, \cdots, I_3$ **do**

  $[\mathbf{U}, \mathbf{S}, \mathbf{V}]$ = SVD $(\hat{\mathcal{A}}(:,:,i_3))$;

  $\hat{\mathcal{U}}(:,:,i_3)$=$\mathbf{U}$, $\hat{\mathcal{S}}(:,:,i_3)$ = $\mathbf{S}$, $\hat{\mathcal{V}}(:,:,i_3)$ = $\mathbf{V}$.

end

$\mathcal{U} \leftarrow$ ifft($\hat{\mathcal{U}}$,[],3), $\mathcal{S} \leftarrow$ ifft($\hat{\mathcal{S}}$,[],3), $\mathcal{V} \leftarrow$ ifft($\hat{\mathcal{V}}$,[],3).

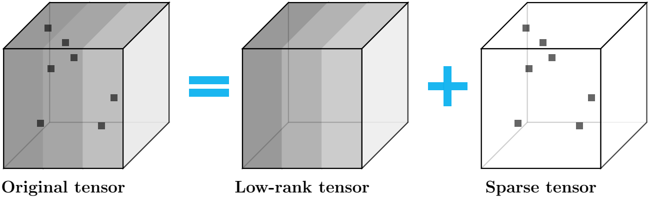
Output: $\mathcal{U}, \mathcal{S}, \mathcal{V}$.

- The TNN of tensor $\mathcal{A}$ is defined as

$$\|\mathcal{A}\|_* = \frac{1}{I_3} \sum_{i_3=1}^{I_3} \|\hat{\mathcal{A}}^{(i_3)}\|_* = \frac{1}{I_3} \sum_{i=1}^{\min(I_1, I_2)} \sum_{i_3=1}^{I_3} \hat{\mathcal{S}}(i, i, i_3). \tag{1}$$

- The ℓ_1-norm of three way tensor $\mathcal{B}$ is $\|\mathcal{B}\|_1 = \sum_{i_1, i_2, i_3} |b_{i_1, i_2, i_3}|$.

Robust Tensor Principal Component Analysis



$$\min_{\mathcal{L},\ \mathcal{E}} \|\mathcal{L}\|_* + \lambda\|\mathcal{E}\|_1, \quad \text{s.t.} \ \ \mathcal{X} = \mathcal{L} + \mathcal{E}, \tag{2}$$

where λ is a regularization parameter, $\|\mathcal{L}\|_*$ denotes the tensor nuclear norm of low-rank tensor $\mathcal{L}$, and $\|\mathcal{E}\|_1$ is the ℓ_1 norm for the sparse tensor.

Robust Block Tensor Principal Component Analysis

Main idea: block the whole tensor into the concatenation of block tensors in the same size.

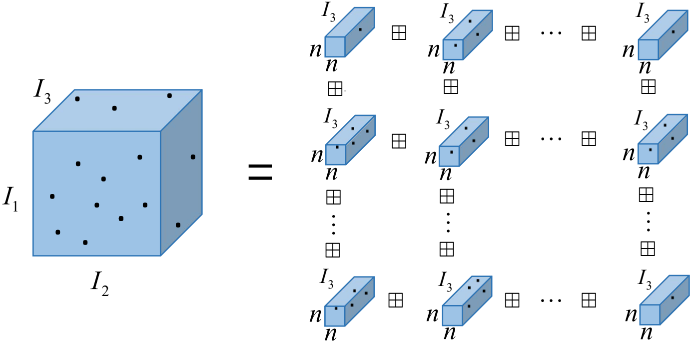


Figure 1: Illustration of the RBTPCA model.

Robust Block Tensor Principal Component Analysis

The proposed RBTPCA method can be formulated into the following convex optimization model:

$$
\begin{aligned}
&\min_{\mathcal{L}_p,\mathcal{E}_p} \sum_{p=1}^{P} (\|\mathcal{L}_p\|_* + \lambda \|\mathcal{E}_p\|_1) \\
&\text{s. t. } \mathcal{X} = \mathcal{L}_1 \boxplus \cdots \boxplus \mathcal{L}_P + \mathcal{E}_1 \boxplus \cdots \boxplus \mathcal{E}_P
\end{aligned}
\tag{3}
$$

- P represents the number of the block tensors decomposed by the whole tensor.
- "$\boxplus$" denotes the concatenation operator of block tensors.
- $\mathcal{L}_p,\ p = 1, 2, \ldots, P$ denotes the block low rank component.
- $\mathcal{E}_p,\ p = 1, 2, \ldots, P$ is the block sparse component.

Illumination Normalization for Face Images

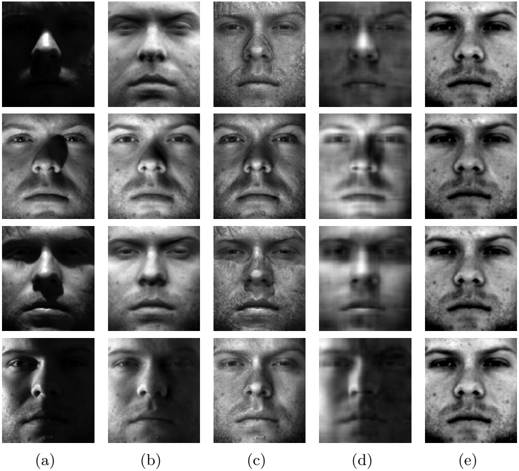
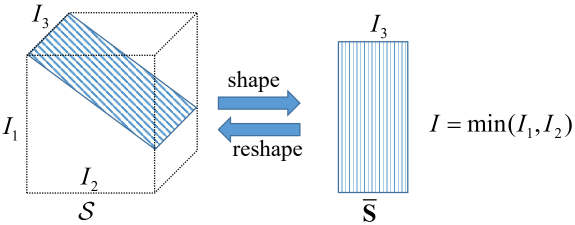


Figure 2: Four methods for removing shadows on face images with size $192 \times 168 \times 64$. (a) original faces with shadows; (b) RPCA; (c) multi-scale low rank decomposition; (d) RTPCA; (e)**RBTPCA**.

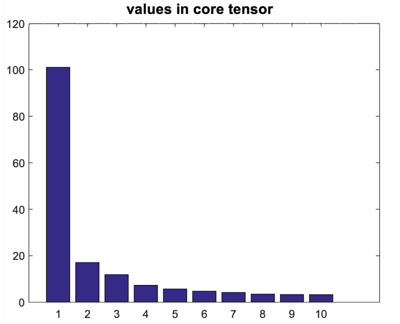
Improved Robust Tensor Principal Component Analysis

Main idea: reshape core singular value tensor along the third mode.

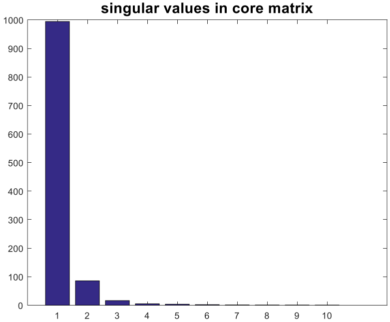


- $\overline{\mathbf{S}}$ denotes the core singular value matrix.
- Can we make core tensor more diagonal?
- Tensor nuclear norm can be presented in another way.

Example : Difference of Singular Values



(a)



(b)

- Tensor data is the surveillance video in hall.
- Values in core tensor decrease slowly, however, singular values for core matrix $\overline{\mathbf{S}}$ decrease rapidly.
- Core tensor has low rank structures.

Improved Robust Tensor Principal Component Analysis

- The improved tensor nuclear norm (ITNN) is defined as follows:

$$\|\mathcal{L}\|_{\mathrm{ITNN}} = \|\mathcal{L}\|_* + \lambda_S \|\overline{\mathbf{S}}\|_* \tag{4}$$

  where λ_S is a parameter to balance the two terms. The additional term $\|\mathbf{S}\|_*$ can additionally exploit low rank information in the third mode.

- The improved robust tensor principal component analysis (IRTPCA) optimization model is formulated as:

$$\min_{\mathcal{L},\ \mathcal{E}} \|\mathcal{L}\|_{\mathrm{ITNN}} + \lambda \|\mathcal{E}\|_1, \quad \text{s.t. } \mathcal{X} = \mathcal{L} + \mathcal{E}. \tag{5}$$

Frequency Component Analysis

- When the FFT is conducted on the third mode of $\mathcal{A}$, different frontal slices represent different frequency components and have vary physical meanings.

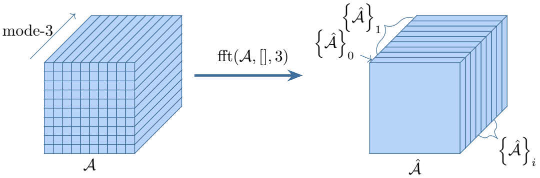


Figure 3: Illustration about the FFT on an order-3 tensor.

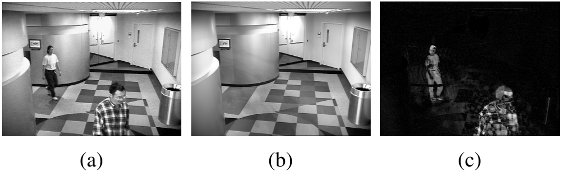


Figure 4: FCA results of a grayscale video with size $320 \times 240 \times 90$. (a) Original; (b) zero frequency component; (c) non-zero frequency components.

Frequency-Weighted Robust Tensor Principal Component Analysis

- To explore the frequency prior knowledge of data, the Frequency-Weighted TNN (FTNN) is defined as follows:

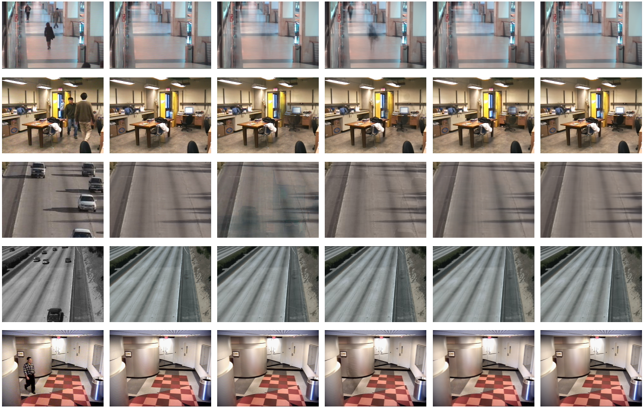
$$\begin{aligned}\|\mathcal{L}\|_{\mathsf{FTNN}} &= \frac{1}{I_3}\left\|\begin{bmatrix}\alpha_1\hat{\mathcal{L}}^{(1)} & & & \\ & \alpha_2\hat{\mathcal{L}}^{(2)} & & \\ & & \ddots & \\ & & & \alpha_{I_3}\hat{\mathcal{L}}^{(I_3)}\end{bmatrix}\right\|_* \\ &= \frac{1}{I_3}\sum_{i_3=1}^{I_3}\alpha_{i_3}\|\hat{\mathcal{L}}^{(i_3)}\|_*, \end{aligned} \tag{6}$$

  where $\alpha_{i_3} \geq 0, i_3 = 1, \cdots, I_3$ is called as frequency weight or filtering coefficient.

- The frequency-weighted robust tensor principal component analysis (FRT-PCA) model can be represented as follows:

$$\min_{\mathcal{L},\ \mathcal{E}} \|\mathcal{L}\|_{\mathsf{FTNN}} + \lambda\|\mathcal{E}\|_1, \quad \text{s. t. } \ \mathcal{X} = \mathcal{L} + \mathcal{E}. \tag{7}$$

Background Modeling for Surveillance Videos



(a) (b) (c) (d) (e) (f)

Figure 5: Recovered background images of 5 example sequences. (a) Original; (b) Ground-truth; (c) RPCA; (d) RTPCA; (e) **IRTPCA**; (f) **FRTPCA**.

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THANK YOU

Any Questions