An L1-L2 Variant of Tubal Nuclear Norm for Guaranteed Tensor Recovery

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Tensor data is almost everywhere!
Tensor Recovery

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But in many applications, only a few noisy observations are available.

How to recover the true tensor from a few noisy observations?

The Observation Model

observation design tensor

\[ y_i = \langle \mathcal{L}^*, \mathcal{X}_i \rangle + \xi_i, \quad i = 1, \cdots, N \]

true tensor noise

To recover \( \mathcal{L}^* \), the tubal nuclear norm based models were proposed.

C. Lu et al. Exact low tubal rank tensor recovery from Gaussian measurements. IJCAI 2018.
Tubal nuclear norm (TNN)

TNN is the averaged nuclear norm of frontal slices after DFT

\[
\|\mathcal{T}\|_\# := \frac{1}{d_3} \sum_{i=1}^{d_3} \|\tilde{T}^{(i)}\|_*
\]

also equal to the scaled nuclear norm of Fourier block diagonal matrix

\[
\|\mathcal{T}\|_\# = \frac{1}{d_3} \|\mathcal{T}\|_*
\]

TNN yields sub-optimal performance due to \textit{biased approximation} of \(\text{rank}(\mathcal{T})\)

Tensor L1-L2 metric for tensor recovery

**L1-L2 metric**

With $\alpha > 0$, define the vector $l_1 - \alpha l_2$ metric

$$\|x\|_{\alpha, 1-2} := \|x\|_1 - \alpha \|x\|_2.$$ 

**A tighter approximation of L0-norm than L1-norm**

**Tensor L1-L2 metric**

$$\|\mathcal{T}\|_{\alpha, \circ-F} := \frac{1}{d_3} \left\| \sigma^{(0)}(\mathbf{T}) \right\|_{\alpha, 1-2}$$

**A tighter approximation of rank(\mathbf{T}) than TNN**

**The proposed estimator**

$$\hat{\mathcal{L}} \in \arg \min_{\mathcal{L}} \|\mathcal{L}\|_{\alpha, \circ-F}$$

s.t. \(\|y - \mathbf{X}(\mathcal{L})\| \leq \tau\)

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Figure 1: Level curves of the $l_1$-norm and $l_1 - \alpha l_2$ metric ($\alpha = 1$).


Bound on estimation error

"How well can we estimate the true tensor?"

Under a new tensor RIP condition, we guarantee stable tensor recovery.

\[ \| \hat{\mathcal{L}} - \mathcal{L}^* \|_F \]

**Theorem 2** (Stable recovery under tm-RIP). Suppose the true tensor \( \mathcal{L}^* \) in Model (4) has multi-rank \( r = (r_1, \cdots, r_{d_3}) \). If there exists a positive integer \( s \) such that

\[
\Phi_{r,s} := 1 - \delta_{2r+s}^x - \frac{\sqrt{4r_1 + 2\alpha^2 d_3}}{\sqrt{s}}(\delta_{2r+s}^x + \delta_{2s}^x) > 0, \tag{8}
\]

where \( s = (s, \cdots, s) \in \mathbb{R}^{d_3} \), then any \( \hat{\mathcal{L}} \) in Eq. (5) obeys

\[
\| \hat{\mathcal{L}} - \mathcal{L}^* \|_F \leq \frac{2\sqrt{1 + \delta_{2r+s}^x}}{\Phi_{r,s}} \tau. \tag{9}
\]
The optimization algorithm

1. Derive a closed-form solution of the proximality operator the metric.

2. Add auxiliary variables for better decoupling, and use ADMM

\[ \hat{\mathcal{L}} \in \arg\min_{\mathcal{L}} \| \mathcal{L} \|_{\alpha, \otimes} - F \]

s.t. \[ \| y - \mathcal{X}(\mathcal{L}) \| \leq \tau \]

auxiliary variables

\[ \min_{\mathcal{L}, \mathcal{K}, \epsilon} \| \mathcal{L} \|_{\alpha, \otimes} - F \]

s.t. \[ \mathcal{K} = \mathcal{L}, \mathcal{X}(\mathcal{K}) + \epsilon = y, \epsilon \in \mathbb{B}_\tau. \]

The proposed metric has promising performance in tensor recovery.